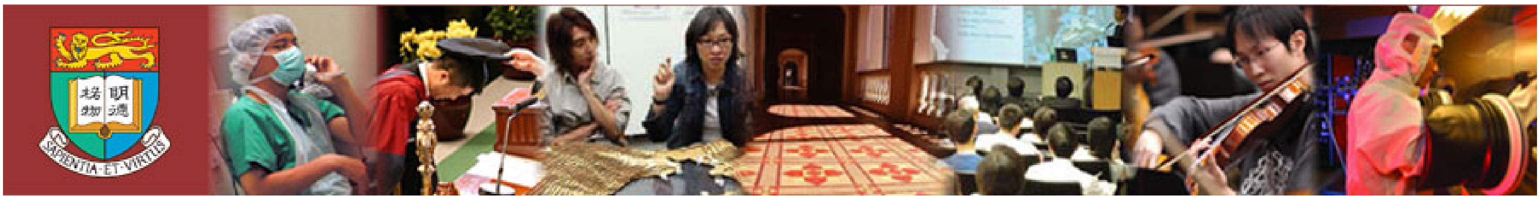

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On the Robust Stability of 2D Mixed Continuous-Discrete-Time Systems with Uncertainty

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Abstract— **This paper addresses the problem of establishing robust exponential stability of 2D mixed continuous-discrete-time systems affected by uncertainty. Specifically, it is supposed that the matrices of the system are polynomial functions of an uncertain vector constrained over a semialgebraic set. First, it is shown that robust exponential stability is equivalent to the existence of a complex Lyapunov functions depending polynomially on the uncertain vector and an additional parameter of degree not greater than a known quantity. Second, a condition for establishing robust exponential stability is proposed via convex optimization by exploiting sums-of-squares (SOS) matrix polynomials. This condition is sufficient for any chosen degree of the complex Lyapunov function candidate, and is also necessary for degrees sufficiently large.**

I. INTRODUCTION

2D mixed continuous-discrete-time systems play a key role in control engineering. For example, they can be found in repetitive processes [23], disturbance propagation in vehicle platoons [10], and irrigation channels [15], [17]. Their study has a long history, with some early works such as [8], [22] introducing basic models, systems theory and stability properties.

Several conditions for establishing exponential stability of 2D systems have been proposed in the literature. For instance, stability of 2D discrete-discrete systems is investigated in [19] through the use of a 2D characteristic polynomial (or more accurately, a multinomial), and in [1] which proposes algebraic necessary and sufficient conditions. Stability of 2D mixed continuous-discrete-time systems is investigated in [24] through the use of Kronecker products, and in [3], [7], [11], [12], [14] via linear matrix inequality (LMI) feasibility tests.

However, the models of 2D mixed continuous-discrete-time systems are generally affected by uncertainty, either because the coefficients of the systems cannot be measured exactly, or because they are subject to changes. This introduces a nontrivial difficulty since one should repeat the existing conditions for addressing the uncertainty-free case for all the admissible values of the uncertainty. Some existing works have derived conditions for robust exponential stability of 2D mixed continuous-discrete-time systems affected by uncertainty, such as [20] which exploits LMIs, however these conditions are only sufficient. It is worth mentioning that necessary and sufficient LMI conditions for robust exponential stability have been proposed in the case of 1D uncertain systems, see for instance [2], [5], [6], [18], [25].

This paper addresses the problem of establishing robust exponential stability of 2D mixed continuous-discrete-time systems affected by uncertainty. Specifically, it is supposed that the matrices of the system are polynomial functions of an uncertain vector constrained over a semialgebraic set. First, it is shown that robust exponential stability is equivalent to the existence of a complex Lyapunov functions depending polynomially on the uncertain vector and an additional parameter of degree not greater than a known quantity. Second, a condition for establishing robust exponential stability is proposed via convex optimization by exploiting sums-of-squares (SOS) matrix polynomials. This condition is sufficient for any chosen degree of the complex Lyapunov function candidate, and is also necessary for degrees sufficiently large. Some numerical examples illustrate the proposed results.

II. PRELIMINARIES

A. Problem Formulation

Notation: $\mathbb{R}, \mathbb{C}$: real and complex number sets; j: imaginary unit, i.e. $j^2 = -1$; I: identity matrix (of size specified by the context); $\Re(A)$, $\Im(A)$: real and imaginary parts of A; $\bar{A}$: complex conjugate of A; A^T, A^H: transpose and complex conjugate transpose of A; $adj(A)$: adjoint of A; $\det(A)$: determinant of A; $trace(A)$: trace of A; $\lambda_i(A)$: i-th eigenvalue of A; $\|A\|_2$: Euclidean norm of A; $|a|$: magnitude of a; Hermitian matrix A: a complex square matrix satisfying $A^H = A$; $\star$: corresponding block in symmetric or Hermitian matrices; $A > 0$, $A \geq 0$: Hermitian positive definite and Hermitian positive semidefinite matrix A.

Let us consider the 2D mixed continuous-discrete-time system with uncertainty described by

$$\left\{ \begin{array}{rcl} \dfrac{d}{dt}x_c(t,k) & = & A_{cc}(p)x_c(t,k) + A_{cd}(p)x_d(t,k) \\ x_d(t,k+1) & = & A_{dc}(p)x_c(t,k) + A_{dd}(p)x_d(t,k) \end{array} \right. \tag{1}$$

where $x_c \in \mathbb{R}^{n_c}$ and $x_d \in \mathbb{R}^{n_d}$ are the continuous and discrete states, respectively, the scalars t and k are independent variables, and $p \in \mathbb{R}^q$ is a time-invariant uncertain vector. It is supposed that p is constrained as

$$p \in \mathcal{P} \tag{2}$$

where $\mathcal{P}$ is the set of admissible uncertainties modeled by

$$\mathcal{P} = \{p \in \mathbb{R}^q : \ a_i(p) \geq 0 \ \forall i = 1\ldots, n_a\} \tag{3}$$

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where $a_i(p)$ $i = 1, \ldots, n_a$, are polynomials. The matrices $A_{cc} : \mathbb{R}^q \to \mathbb{R}^{n_c \times n_c}$, $A_{cd} : \mathbb{R}^q \to \mathbb{R}^{n_c \times n_d}$, $A_{dc} : \mathbb{R}^q \to \mathbb{R}^{n_d \times n_c}$ and $A_{dd} : \mathbb{R}^q \to \mathbb{R}^{n_d \times n_d}$ are polynomial functions of degree not greater than d_A.

Extending the classical definition of exponential stability of 2D mixed continuous-discrete-time systems (see, e.g., [19], [27]), we say that the system (1)–(3) is robustly exponentially stable if there exist $\beta, \delta \in \mathbb{R}$ such that

$$\left\| \begin{pmatrix} x_c(t,k) \\ x_d(t,k) \end{pmatrix} \right\|_2 \leq \beta \varrho e^{-\delta \min\{t,k\}} \tag{4}$$

for all $t \geq 0$ and $k \geq 0$, for all initial conditions $x_c(0,k)$ and $x_d(t,0)$, and for all $p \in \mathcal{P}$, where

$$\begin{gathered} \varrho = \max\{\varrho_1, \varrho_2\} \\ \varrho_1 = \sup_{t \geq 0} \|x_d(t,0)\|_2\,, \quad \varrho_2 = \sup_{k \geq 0} \|x_c(0,k)\|_2\,. \end{gathered} \tag{5}$$

Problem. The problem addressed in this paper consists of establishing whether (1)–(3) is robustly exponentially stable. □

B. SOS Matrix Polynomials

Here we provide some information about establishing whether a matrix polynomial is SOS via an LMI feasibility test. For reasons that will become clear in the next section, let us consider a symmetric matrix polynomial $J : \mathbb{R} \times \mathbb{R}^q \to \mathbb{R}^{2n_d \times 2n_d}$.

The matrix polynomial $J(\omega,p)$, $\omega \in \mathbb{R}$ and $p \in \mathbb{R}^q$, is said to be SOS if there exist matrix polynomials $J_i : \mathbb{R} \times \mathbb{R}^q \to \mathbb{R}^{2n_d \times 2n_d}$, $i = 1, \ldots, k$, such that

$$J(\omega,p) = \sum_{i=1}^{k} J_i(\omega,p)^T J_i(\omega,p). \tag{6}$$

A necessary and sufficient condition for establishing whether $J(\omega,p)$ is SOS can be obtained via an LMI feasibility test.

Indeed, $J(\omega,p)$ can be expressed as

$$J(\omega,p) = \left(b(\omega,p) \otimes I\right)^T \left(K + L(\alpha)\right) \left(b(\omega,p) \otimes I\right) \tag{7}$$

where $b(\omega,p) \in \mathbb{R}^c$ is a vector whose entries are the monomials in ω and p of degree less than or equal to d, and c is the number of these monomials given by

$$c = \frac{(q+1+d)!}{(q+1)!d!}, \tag{8}$$

$K \in \mathbb{R}^{2cn_d \times 2cn_d}$, $K = K^T$, satisfies

$$J(\omega,p) = \left(b(\omega,p) \otimes I\right)^T K \left(b(\omega,p) \otimes I\right), \tag{9}$$

$L : \mathbb{R}^\tau \in \mathbb{R}^{2cn_d \times 2cn_d}$ is a linear parametrization of the linear subspace

$$\mathcal{L} = \left\{ L = L^T : \ \left(b(\omega,p) \otimes I\right)^T L \left(b(\omega,p) \otimes I\right) = 0 \right\} \tag{10}$$

and $\alpha \in \mathbb{R}^\tau$ is a free vector. The quantity τ is the dimension of $\mathcal{L}$ given by whose dimension is given by

$$\tau = n_d \left(c\left(2cn_d + 1\right) - \left(2n_d + 1\right) \frac{(q+1+2d)!}{(q+1)!(2d)!} \right). \tag{11}$$

The representation (7) is known as square matrix representation (SMR) [5] and extends the Gram matrix method for (scalar) polynomials to the matrix case. One has that $J(\omega,p)$ is SOS if and only if there exists α satisfying the LMI

$$K + L(\alpha) \geq 0. \tag{12}$$

See also [4], [13], [16], [21], [25] and references therein for details on SOS matrix polynomials.

III. ROBUST EXPONENTIAL STABILITY

In this section we address the problem of establishing whether (1)–(3) is robustly exponentially stable.

Let us start by observing that, for the case of 2D mixed continuous-discrete-time systems without uncertainty, a necessary condition for exponential stability is that the matrices A_{cc} and A_{dd} are Hurwitz and Schur, respectively. In particular, we say that A_{cc} is Hurwitz if

$$\Re\left(\lambda_i(A_{cc})\right) < 0 \quad \forall i = 1, \ldots, n_c \tag{13}$$

and we say that A_{dd} is Schur if

$$|\lambda_i(A_{dd})| < 1 \quad \forall i = 1, \ldots, n_d. \tag{14}$$

This means that, without loss of generality, we can introduce the following assumption, which can be checked with existing methods such as [2], [5], [6], [18], [25].

Assumption 1. The matrices $A_{cc}(p)$ and $A_{dd}(p)$ are Hurwitz and Schur, respectively, for all $p \in \mathcal{P}$. □

Let us denote with $X_d(s,k)$ the Laplace transform of $x_d(t,k)$, where $s \in \mathbb{C}$. For null initial conditions, (1) can be rewritten as

$$X_d(s,k+1) = F(s,p)X_d(s,k) \tag{15}$$

where $F : \mathbb{C} \times \mathbb{R}^q \to \mathbb{C}^{n_d \times n_d}$ is given by

$$F(s,p) = A_{dc}(p)(sI - A_{cc}(p))^{-1}A_{cd}(p) + A_{dd}(p). \tag{16}$$

Let us express $F(s,p)$ as

$$F(s,p) = \frac{G(s,p)}{g(s,p)} \tag{17}$$

where $G : \mathbb{C} \times \mathbb{R}^q \to \mathbb{C}^{n_d \times n_d}$ is a matrix polynomial, and $g : \mathbb{C} \times \mathbb{R}^q \to \mathbb{C}$ is the polynomial given by

$$g(s,p) = \det(sI - A_{cc}(p)). \tag{18}$$

The following result directly follows from the definition of robust exponential stability and the literature (see, e.g., [9]).

Lemma 1: The system (1)–(3) is robustly exponentially stable if and only if

$$|\lambda_i(F(j\omega,p))| < 1 \quad \forall i = 1, \ldots, n_d \ \forall \omega \in \mathbb{R} \ \forall p \in \mathcal{P}. \tag{19}$$

The first contribution of this paper is to derive a sufficient condition for robust exponential stability of the system

(1)–(3) based on the existence of a complex Lyapunov function with polynomial dependence on ω and p, and to show that this condition is also necessary whenever $\mathcal{P}$ is compact, hence providing an upper bound on the minimum degree of such dependency as explained in the following result.

Theorem 1: The system (1)–(3) is robustly exponentially stable if there exist a scalar $\zeta > 0$ and a Hermitian matrix polynomial $V : \mathbb{R} \times \mathbb{R}^q \to \mathbb{C}^{n_d \times n_d}$ such that

$$\left.\begin{array}{rcl} V(\omega,p) - \zeta I & \geq & 0 \\ W(\omega,p) - \zeta \left|g(j\omega,p)\right|^2 I & \geq & 0 \end{array}\right\} \quad \forall \omega \in \mathbb{R} \ \forall p \in \mathcal{P} \tag{20}$$

where

$$W(\omega,p) = \left|g(j\omega,p)\right|^2 V(\omega,p) - G(j\omega,p)^H V(\omega,p) G(j\omega,p). \tag{21}$$

Moreover, if $\mathcal{P}$ is compact, this condition is not only sufficient but also necessary, and the degree of $V(\omega,p)$ can be chosen not greater than

$$2\mu = 2\max\{n_c n_d^2, d_A((n_c+1)n_d^2 - 1)\}. \tag{22}$$

Proof. "$\Leftarrow$" Suppose that there exists a scalar $\zeta > 0$ and a Hermitian matrix polynomial $V : \mathbb{R} \times \mathbb{R}^q \to \mathbb{C}^{n_d \times n_d}$ such that (20) holds. From Assumption 1 it follows that

$$g(j\omega,p) \neq 0 \quad \forall \omega \in \mathbb{R} \ \forall p \in \mathcal{P} \tag{23}$$

which implies

$$\left.\begin{array}{rcl} V(\omega,p) - \zeta I & \geq & 0 \\ V(\omega,p) - F(j\omega,p)^H & & \\ \cdot V(\omega,p) F(j\omega,p) - \zeta I & \geq & 0 \end{array}\right\} \quad \forall \omega \in \mathbb{R} \ \forall p \in \mathcal{P} \tag{24}$$

and, hence, (19) holds. Consequently, from Lemma 1 we conclude that the system (1)–(3) is robustly exponentially stable.

"$\Rightarrow$" Suppose that the system (1)–(3) is robustly exponentially stable and that $\mathcal{P}$ is compact. From Lemma 1 one has that (19) holds. From Assumption 1 it follows that the discrete Lyapunov equation

$$V(\omega,p) - F(j\omega,p)^H V(\omega,p) F(j\omega,p) = I, \quad p \in \mathcal{P} \tag{25}$$

has a unique solution $V(\omega,p)$ which satisfies

$$V(\omega,p) \geq \varepsilon_1 \quad \forall \omega \in \mathbb{R} \ \forall p \in \mathcal{P} \tag{26}$$

for some $\varepsilon_1 > 0$. The discrete Lyapunov equation (25) can be rewritten as

$$E(\omega,p) v(\omega,p) = q \tag{27}$$

where $v(\omega,p)$ and q are $n_d^2 \times 1$ vectors that gather the free entries of $V(\omega,p)$ and I (i.e., the entries in the upper triangular part of these matrices, since those in the lower triangular part are automatically defined as the matrices are Hermitian), and $E(\omega,p) \in \mathbb{R}^{n_d^2 \times n_d^2}$ satisfies

$$\left|\det(E(\omega,p))\right| \geq \varepsilon_2 \quad \forall \omega \in \mathbb{R} \ \forall p \in \mathcal{P} \tag{28}$$

for some $\varepsilon_2 > 0$ due to Assumption 1 and since the solution of the discrete Lyapunov equation (25) is unique. Hence,

$$v(\omega,p) = E(\omega,p)^{-1} q. \tag{29}$$

Let us observe that $E(\omega,p)$ can be written as

$$E(\omega,p) = \frac{E_N(\omega,p)}{\left|g(j\omega,p)\right|^2} \tag{30}$$

where $E_N(\omega,p)$ is a matrix polynomial. In particular, since $\left|g(j\omega)\right|^2$ is a polynomial of degree not greater than

$$2d_g = 2n_c \max\{1, d_A\}, \tag{31}$$

it follows that $E_N(\omega,p)$ is a matrix polynomial of degree not greater than

$$2d_E = 2\max\{n_c, d_A(n_c+1)\}. \tag{32}$$

Taking into account (23), one obtains

$$\left|\det(E_N(\omega,p))\right| \geq \varepsilon_3 \quad \forall \omega \in \mathbb{R} \ \forall p \in \mathcal{P} \tag{33}$$

for some $\varepsilon_3 > 0$. Hence, the inverse of $E(\omega,p)$ does exist and is given by

$$E(\omega,p)^{-1} = \left|g(j\omega,p)\right|^2 \frac{adj(E_N(\omega,p))}{\det(E_N(\omega,p))} \tag{34}$$

where $adj(E_N(\omega,p))$ is a matrix polynomial of degree not greater than $2d_E(n_d^2 - 1)$ and $\det(E_N(\omega))$ is a polynomial of degree not greater than $2d_E n_d^2$. Hence,

$$\begin{array}{rcl} v(\omega,p) & = & \left|g(j\omega,p)\right|^2 \dfrac{adj(E_N(\omega,p))}{\det(E_N(\omega,p))} q \\ & = & \dfrac{v_N(\omega,p)}{\det(E_N(\omega,p))} \end{array} \tag{35}$$

where $v_N(\omega,p)$ is a vector polynomial of degree not greater than 2μ. Equation (35) shows that the solution $V(\omega,p)$ of the discrete Lyapunov equation (25) is a matrix rational function.

Next, let us define

$$c(\omega,p) = sgn(\det(E_N(0,p_0))) \det(E_N(\omega,p)) \tag{36}$$

where p_0 is arbitrary in $\mathcal{P}$. Since $E_N(\omega,p)$ is a matrix polynomial, (33) implies that

$$c(\omega,p) \geq \varepsilon_4 \quad \forall \omega \in \mathbb{R} \ \forall p \in \mathcal{P} \tag{37}$$

for some $\varepsilon_4 > 0$. Let us define

$$\hat{V}(\omega,p) = c(\omega,p) V(\omega,p). \tag{38}$$

Due to Assumption 1, it follows that

$$\left.\begin{array}{rcl} \hat{V}(\omega,p) - \varepsilon_1\varepsilon_4 I & \geq & 0 \\ \left|g(j\omega,p)\right|^2 \hat{V}(\omega,p) & & \\ -G(j\omega,p)^H \hat{V}(\omega,p) G(j\omega,p) & & \\ -c(\omega,p)\left|g(j\omega,p)\right|^2 I & = & 0 \end{array}\right\} \quad \forall \omega \in \mathbb{R} \ \forall p \in \mathcal{P}. \tag{39}$$

Hence, (20) holds with $V(\omega,p)$ replaced by $\hat{V}(\omega,p)$, which is a matrix polynomial of degree not greater than 2μ, and ζ given by

$$\zeta = \min\{\varepsilon_1\varepsilon_4, \varepsilon_4\}. \tag{40}$$

$\square$

Theorem 1 states that the robust exponential stability of (1)–(3) can be established if there exist a scalar $\zeta > 0$ and a Hermitian matrix polynomial $V(\omega, p)$ satisfying (20). This matrix polynomial defines a complex (quadratic) Lyapunov function with polynomial dependence on ω and p. Moreover, Theorem 1 also states that this condition is not only sufficient but also necessary whenever $\mathcal{P}$ is compact, and provides an upper bound on the minimum degree of $V(\omega, p)$.

Results analogous to Theorem 1 have been proposed in [2], [5], [6], [18], [25], where the existence of a suitable parameter-dependent Lyapunov function is proved for the case of 1D uncertain systems.

At this point, the problem is how to check whether (20) holds. To this end, let us define the matrix function $\Phi : \mathbb{C}^{n_d \times n_d} \to \mathbb{R}^{2n_d \times 2n_d}$ as

$$\Phi(S) = \begin{pmatrix} S_R & S_I \\ * & S_R \end{pmatrix} \tag{41}$$

where $S_R, S_I \in \mathbb{R}^{n_d \times n_d}$ are the real and imaginary parts of S, i.e., $S = S_R + jS_I$. Let us observe that

$$S \text{ is Hermitian} \iff \Phi(S) = \Phi(S)^T. \tag{42}$$

The following result provides the second contribution of this paper, which is an LMI condition for establishing the robust exponential stability of (1)–(3).

Theorem 2: The system (1)–(3) is robustly exponentially stable if there exist a scalar $\zeta > 0$ and Hermitian matrix polynomials $V, S_i, T_i : \mathbb{R} \times \mathbb{R}^q \to \mathbb{C}^{n_d \times n_d}$, $i = 1, \ldots, n_a$, such that

$$\begin{cases} \Phi\left(X(\omega, p)\right) \text{ is SOS} \\ \Phi\left(Y(\omega, p)\right) \text{ is SOS} \\ \Phi\left(S_i(\omega, p)\right) \text{ is SOS } \forall i = 1, \ldots, n_a \\ \Phi\left(T_i(\omega, p)\right) \text{ is SOS } \forall i = 1, \ldots, n_a \end{cases} \tag{43}$$

where

$$\begin{array}{rcl} X(\omega, p) & = & V(\omega, p) - \zeta I - \displaystyle\sum_{i=1}^{n_a} a_i(p) S_i(\omega, p) \\ Y(\omega, p) & = & W(\omega, p) - \zeta \left|g(j\omega, p)\right|^2 I \\ & & - \displaystyle\sum_{i=1}^{n_a} a_i(p) T_i(\omega, p). \end{array} \tag{44}$$

Proof. Suppose that there exist a scalar $\zeta > 0$ and Hermitian matrix polynomials $V, S_i, T_i : \mathbb{R} \times \mathbb{R}^q \to \mathbb{C}^{n_d \times n_d}$, $i = 1, \ldots, n_a$, such that (43) holds. From the definition of SOS matrix polynomials in Section II-B, the first constraint in (43) implies that

$$\Phi\left(X(\omega, p)\right) \geq 0 \quad \forall \omega \in \mathbb{R} \ \forall p \in \mathbb{R}^q. \tag{45}$$

From (41) it follows that

$$X(\omega, p) \geq 0 \quad \forall \omega \in \mathbb{R} \ \forall p \in \mathbb{R}^q. \tag{46}$$

Similarly, one obtains that $Y(\omega, p)$, $S_i(\omega, p)$ and $T_i(\omega, p)$, $i = 1, \ldots, n_a$, are positive semidefinite for all $\omega \in \mathbb{R}$ and for all $p \in \mathbb{R}^q$.

Next, let ω and p be arbitrary in $\mathbb{R}$ and $\mathcal{P}$, respectively. The positive semidefiniteness of $X(\omega, p)$ and $S_i(\omega, p)$, $i = 1, \ldots, n_a$, implies that

$$\begin{array}{rcl} 0 & \leq & X(\omega, p) \\ & = & V(\omega, p) - \zeta I - \sum_{i=1}^{n_a} a_i(p) S_i(\omega, p) \\ & \leq & V(\omega, p) - \zeta I \end{array} \tag{47}$$

since $a_i(p) \geq 0$, $i = 1, \ldots, n_a$. This means that $V(\omega, p) - \zeta I$ is positive semidefinite for all $\omega \in \mathbb{R}$ and for all $p \in \mathbb{R}^q$. Analogously, one proves that $W(\omega, p) - \zeta \left|g(j\omega, p)\right|^2 I$ is positive semidefinite for all $\omega \in \mathbb{R}$ and for all $p \in \mathbb{R}^q$. Hence, (20) holds. Lastly, ζ is positive, and therefore we can conclude from Theorem 1 that the system (1)–(3) is robustly exponentially stable. $\square$

Theorem 2 provides a condition based on SOS matrix polynomials for establishing the robust exponential stability of (1)–(3). Since establishing whether a matrix polynomials is SOS is equivalent to an LMI as explained in Section II-B, and since the matrix polynomials $X(\omega, p)$ and $Y(\omega, p)$ are affine linear matrix functions of the variables $V(\omega, p)$, $S_i(\omega, p)$, $T_i(\omega, p)$ and ϵ, it follows that the condition (43) amounts to checking whether a system of LMIs is feasible, which is a convex optimization problem.

For any chosen degrees of the matrix polynomials $V(\omega, p)$, $S_i(\omega, p)$ and $T_i(\omega, p)$, the LMI condition provided by Theorem 2 is sufficient to establish the robust exponential stability of (1)–(3). The conservatism of this condition can be decreased by increasing the degrees of these polynomials.

A simple way to choose the degrees of the matrix polynomials $V(\omega, p)$, $S_i(\omega, p)$ and $T_i(\omega, p)$ is the following. First, one arbitrarily chooses the degree of the matrix polynomial $V(\omega, p)$, which defines the candidate complex Lyapunov function. Then, one chooses the degree of the matrix polynomials $S_i(\omega, p)$ and $T_i(\omega, p)$ as the largest degrees for which the matrix polynomials $X(\omega, p)$ and $Y(\omega, p)$ have their minimum degrees. This choice will be adopted hereafter.

In order to quantify the feasibility of (43), we introduce the index

$$\begin{array}{rcl} \zeta^* & = & \displaystyle\sup_{\substack{\zeta \in \mathbb{R} \\ V, S_i, T_i : \mathbb{R} \times \mathbb{R}^q \to \mathbb{C}^{n_d \times n_d}}} \zeta \\ & & \text{s.t.} \begin{cases} (43) \text{ holds} \\ trace(V(\omega_0, p_0)) = 10 \end{cases} \end{array} \tag{48}$$

where ω_0 and p_0 are arbitrarily chosen values in $\mathbb{R}$ and $\mathcal{P}$, respectively. This index quantifies feasibility of the condition provided by Theorem 2, in particular

$$(43) \text{ holds} \iff \zeta^* > 0. \tag{49}$$

The optimization problem (48), which amounts to minimizing a linear function subject to SOS constraints and linear equations, is a convex optimization problem, in particular a semidefinite program.

The next result states that the LMI condition provided by Theorem 2 is not only sufficient but also necessary under some conditions on $\mathcal{P}$.

Theorem 3: Suppose that the system (1)–(3) is robustly exponentially stable and that $\mathcal{P}$ is compact. Suppose also that $a_i(p)$, $i = 1, \ldots, n_a$, have even degree, and that their highest degree homogeneous parts have no common zeros except 0. Then, there exist a scalar $\zeta > 0$ and Hermitian matrix polynomials $V, S_i, T_i : \mathbb{R} \times \mathbb{R}^q \to \mathbb{C}^{n_d \times n_d}$, $i = 1, \ldots, n_a$, such that (43) holds.

Proof. Suppose that the system (1)–(3) is robustly exponentially stable and that $\mathcal{P}$ is compact. From Theorem 1 it follows that there exist a scalar $\zeta > 0$ and a Hermitian matrix polynomial $V : \mathbb{R} \times \mathbb{R}^q \to \mathbb{C}^{n_d \times n_d}$ of degree not greater than 2μ such that (20) holds. Let us consider the first constraint in (20). This is equivalent to

$$\Phi\left(V(\omega, p) - \zeta I\right) \geq 0 \quad \forall \omega \in \mathbb{R} \; \forall p \in \mathcal{P}. \tag{50}$$

Similarly to Section II-B where the SMR of a matrix polynomial in ω and p has been described, let us introduce the parametrized SMR of $\Phi\left(V(\omega, p) - \zeta I\right)$ with respect to p as

$$\begin{aligned}\Phi\left(V(\omega, p) - \zeta I\right) \;\; = \;\; & \left(\hat{b}(\omega) \otimes I\right)^T \left(\hat{K}(p) + \hat{L}(\hat{\alpha}(p))\right) \\ & \cdot \left(\hat{b}(\omega) \otimes I\right)\end{aligned} \tag{51}$$

where $\hat{b}(\omega)$ is a vector whose entries are the monomials in ω of degree less than or equal to μ, and $\hat{K}(p) + \hat{L}(\hat{\alpha}(p))$ is a symmetric matrix function. In particular, $\hat{K}(p)$ is a symmetric matrix polynomial of degree not greater than 2μ, and $\hat{L}(\hat{\alpha}(p))$ is a linear matrix function of $\hat{\alpha}(p)$, which is an arbitrary vector function of suitable size. Since $\Phi\left(V(\omega, p) - \zeta I\right)$ is a symmetric matrix polynomial and ω is a scalar, (50) is equivalent to the existence of a scalar $\hat{\zeta} > 0$ and a vector function $\hat{\alpha}(p)$ such that (see for instance [4])

$$\hat{K}(p) + \hat{L}(\hat{\alpha}(p)) - \hat{\zeta} I \geq 0 \quad \forall p \in \mathcal{P}. \tag{52}$$

Since $\Phi\left(V(\omega, p) - \zeta I\right)$ is continuous with respect to p and $\mathcal{P}$ is compact, there exists a scalar $\tilde{\zeta} > 0$ and a polynomial vector function $\tilde{\alpha}(p)$ such that

$$\hat{K}(p) + \hat{L}(\tilde{\alpha}(p)) - \tilde{\zeta} I \geq 0 \quad \forall p \in \mathcal{P}. \tag{53}$$

Since $\hat{K}(p) + \hat{L}(\tilde{\alpha}(p)) - \tilde{\zeta} I$ is a symmetric matrix polynomial and due to the assumptions on the polynomials $a_i(p)$, it follows that there exist a scalar $\beta > 0$ and SOS matrix polynomials $\hat{d}(p)$ and $\hat{S}_i(p)$, $i = 1, \ldots, n_a$, of suitable size such that (see for instance [4])

$$\hat{X}(p) = (1 + \hat{d}(p)) \left(\hat{K}(p) + \hat{L}(\tilde{\alpha}(p)) - \beta I\right) - \sum_{i=1}^{n_a} a_i(p) \hat{S}_i(p) \tag{54}$$

is SOS. Hence,

$$\tilde{X}(\omega, p) = \left(\hat{b}(\omega) \otimes I\right)^T \hat{X}(p) \left(\hat{b}(\omega) \otimes I\right) \tag{55}$$

is SOS. Routine calculations show that

$$\begin{aligned}\tilde{X}(\omega, p) \;\; = \;\; & \Phi\Big((1 + \hat{d}(p)) \left(V(\omega, p) - \beta I\right) \\ & - \sum_{i=1}^{n_a} a_i(p) \tilde{S}_i(\omega, p) \Big)\end{aligned} \tag{56}$$

where $\tilde{S}_i(\omega, p)$ are SOS matrix polynomials similarly obtained from $\hat{S}_i(p)$, $i = 1, \ldots, n_a$. Hence, the first and the third constraints in (43) hold with ζ, $V(\omega, p)$ and $S_i(\omega, p)$ replaced by β, $(1 + \hat{d}(p))V(\omega, p)$ and $\tilde{S}_i(\omega, p)$, respectively. Similarly, one proves that also the second and the fourth constraints in (43) hold. □

IV. EXAMPLES

In this section we present two illustrative examples. The optimization problem (48) is solved with the toolbox SeDuMi [26] for Matlab. Assumption 1 holds in both examples.

A. Example 1

Let us consider (1)–(3) with

$$A_{cc}(p) = \begin{pmatrix} 0 & 1 \\ -1 & -2 - p_1 \end{pmatrix}, \; A_{cd}(p) = \begin{pmatrix} 0 & p_2 \\ 0.3 & 0 \end{pmatrix}$$

$$A_{dc}(p) = \begin{pmatrix} -0.8 & 0.5 \\ 0 & 0.4 \end{pmatrix}, \; A_{dd}(p) = \begin{pmatrix} 0 & -0.9 \\ 0 & -0.5 \end{pmatrix}$$

$$\mathcal{P} = \left\{ p \in \mathbb{R}^2 : \; p_1^2 + p_2^2 \leq 1 \right\}.$$

Hence, it turns out that $n_c = n_d = q = 2$.

We express $\mathcal{P}$ as in (3) with $a(p) = 1 - p_1^2 - p_2^2$. We solve (48) by using candidates Hermitian matrix polynomials $V(\omega, p)$ of degree 0, and we find

$$\zeta^* = 0.089$$

which proves robust exponential stability according to Theorem 2. In particular, the found $V(\omega, p)$ (which defines a common quadratic Lyapunov function) is

$$V(\omega, p) = \begin{pmatrix} 0.148 & -0.404 \\ \star & 9.674 \end{pmatrix}.$$

B. Example 2

Let us consider (1)–(3) with

$$A_{cc}(p) = \begin{pmatrix} 1 & 4 \\ -1 & -2 + 0.2p \end{pmatrix}, \; A_{cd}(p) = \begin{pmatrix} 0.6 & 0.4 \\ 0 & -0.5 - 0.2p \end{pmatrix}$$

$$A_{dc}(p) = \begin{pmatrix} 0 & 1.7 \\ -0.6p & 0 \end{pmatrix}, \; A_{dd}(p) = \begin{pmatrix} 0.5 & 0.3 \\ 0 & 0.2 \end{pmatrix}$$

$$\mathcal{P} = [-1, 1].$$

Hence, it turns out that $n_c = n_d = 2$ and $q = 1$.

We express $\mathcal{P}$ as in (3) with $a(p) = 1 - p^2$. We solve (48) by using candidate Hermitian matrix polynomials $V(\omega, p)$ of degree 0, and we find

$$\zeta^* = -5.253$$

which does not prove robust exponential stability (this suggests that the system does not admit a common quadratic

Lyapunov function). Hence, we use candidate Hermitian matrix polynomials $V(\omega, p)$ of degree 2, finding

$$\zeta^* = 0.376$$

which proves robust exponential stability according to Theorem 2. In particular, the found $V(\omega, p)$ (which defines a complex quadratic Lyapunov function with quadratic dependence on ω and p) is $V(\omega, p) = \Re(V(\omega, p)) + j\Im(V(\omega, p))$ where

$$\Re(V(\omega, p)) = \begin{pmatrix} 3.073 - 10.319p + 10.559p^2 + 2.206\omega^2 & 1.288 - 3.715p + 0.906p^2 - 1.729\omega^2 \\ * & 1.599 + 0.337p + 5.024p^2 + 2.371\omega^2 \end{pmatrix}$$

$$j\Im(V(\omega, p)) = \begin{pmatrix} 0 & j(3.118\omega - 0.109p\omega) \\ * & 0 \end{pmatrix}.$$

V. CONCLUSION

It has been shown that robust exponential stability of 2D mixed continuous-discrete-time systems affected by uncertainty is equivalent to the existence of a complex Lyapunov functions depending polynomially on the uncertain vector and an additional parameter of degree not greater than a known quantity. Also, a condition for establishing robust exponential stability has been proposed via convex optimization by exploiting SOS matrix polynomials. This condition is sufficient for any chosen degree of the complex Lyapunov function candidate, and is also necessary for degrees sufficiently large.

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